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Trentelman, Harry L; Willems, Jan C.

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# CONTROL IN A BEHAVIORAL SETTING

Harry L. Trentelman and Jan C. Willems  
Mathematics Institute, University of Groningen  
P.O. Box 800  
9700 AV Groningen  
The Netherlands

Email: h.l.trentelman@math.rug.nl; j.c.willems@math.rug.nl

## 1 Introduction

Present day control theory is very much centered around the problem of designing a feedback loop around a given system in such a way that in the closed loop system certain design specifications are satisfied. The plant under consideration typically has control inputs, exogenous inputs, measured outputs, and exogenous outputs. The controller to be designed should take the measured outputs of the system as its inputs, and should, on the basis of these inputs, generate control inputs for the plant. These controllers should be designed in such a way that the resulting closed loop system meets the specifications. The above general scheme of approaching control design problems has been called *the intelligent control paradigm* (see [4]).

One of the main features of *the behavioral approach* as a foundational framework for the theory of dynamical systems is that it does not take the input/output structure as the starting point for describing systems in interaction with their environment. Instead, a mathematical model is simply viewed as any relation among variables. In the dynamic case this relation constrains the time-evolution which a set of variables can take. The collection of time trajectories which the model declares possible is called the *behavior* of the dynamical system.

This behavior, hence a set of time functions, can be specified in many different ways. Often, in problem fields as mechanical engineering or electrical engineering, the behavior will be given as the solution set of a system of differential equations, often called *the behavioral equations*. It is our conviction that, in such cases, it is more natural to view controller design as the problem of designing for a given plant an additional set of 'laws' that the variables appearing in the system should obey. More specifically, if a plant is modelled as a set of 'behavioral equations', then the controller design question is to invent an additional set of equations involving the signals appearing in the system, in such a way that the 'controlled system' (i.e., the system consisting of those signals that are compatible with both sets of equations) satisfies the given control specifications, see e.g. [4], [1], [6], [7].

In this note we will explain our new view of control, and

address some issues that come up in developing a theory of control in a behavioral setting. We will also give a few examples of control problems in behavioral setting.

## 2 Control in a behavioral setting

We will first briefly recall our view of control in the context of the behavioral approach to dynamical systems. A dynamical system is a triple,  $\Sigma = (T, W, \mathfrak{B})$  with  $T \subset \mathbb{R}$  the *time axis*,  $W$  a set called the *signal space*, and  $\mathfrak{B} \subset W^T$  the *behavior*. The behavior consists of functions  $w : T \rightarrow W$ . The variable  $w$  is called the *manifest variable* of the system.

If  $\Sigma_1 = (T, W, \mathfrak{B}_1)$  and  $\Sigma_2 = (T, W, \mathfrak{B}_2)$  are two dynamical systems with the same time axis and the same signal space, then the *interconnection* of  $\Sigma_1$  and  $\Sigma_2$ , denoted as  $\Sigma_1 \wedge \Sigma_2$ , is defined as  $\Sigma_1 \wedge \Sigma_2 := (T, W, \mathfrak{B}_1 \cap \mathfrak{B}_2)$ . Thus the behavior of  $\Sigma_1 \wedge \Sigma_2$  consists simply of those trajectories  $w : T \rightarrow W$  which are compatible with both the laws of  $\Sigma_1$  (i.e.,  $w$  belongs to  $\mathfrak{B}_1$ ) and of  $\Sigma_2$  (i.e.,  $w$  belongs to  $\mathfrak{B}_2$ ).

Of course, in most applications, systems are interconnected only through certain terminals and not along others. This situation can easily be incorporated in the definition of interconnection as follows. Assume  $\Sigma_1 = (T, W_1 \times C, \mathfrak{B}_1)$  and  $\Sigma_2 = (T, C \times W_2, \mathfrak{B}_2)$  with their interconnection leading to  $\Sigma_1 \wedge \Sigma_2 = (T, W_1 \times C \times W_2, \mathfrak{B})$  with  $\mathfrak{B} = \{(w_1, c, w_2) : T \rightarrow W_1 \times C \times W_2 \mid (w_1, c) \in \mathfrak{B}_1 \text{ and } (c, w_2) \in \mathfrak{B}_2\}$ .

By redefining  $\Sigma_1$  to  $\hat{\Sigma}_1 = (T, W_1 \times C \times W_2, \hat{\mathfrak{B}}_1)$  with  $\hat{\mathfrak{B}}_1 = \mathfrak{B}_1 \times W_2^T$ , and  $\Sigma_2$  to  $\hat{\Sigma}_2 = (T, W_1 \times C \times W_2, \hat{\mathfrak{B}}_2)$  with  $\hat{\mathfrak{B}}_2 = W_1^T \times \mathfrak{B}_2$ , it is easily seen that this interconnection now becomes a special case of our general definition. Note that the definition of the behavior of  $\hat{\Sigma}_1$  leaves the variables  $w_2$  free, while that of  $\hat{\Sigma}_2$  leaves the variables  $w_1$  free. The variable  $c$  through which the interconnection is established is called the *interconnection variable*. The space  $C$  in which  $c$  takes its values in called *the interconnection space*.

In many interconnections, it is natural to suppress the interconnection variable  $c$  after interconnection, yielding  $\Sigma_1 \wedge \Sigma_2 = (T, W_1 \times W_2, \mathfrak{B})$  with  $\mathfrak{B} = \{(w_1, w_2) : T \rightarrow$

$W_1 \times W_2 \mid \exists c : T \rightarrow C$  such that  $(w_1, c) \in \mathfrak{B}_1$  and  $(c, w_2) \in \mathfrak{B}_2$ . This situation can be formalized using *manifest* and *latent* variables, one of the central features of the behavioral approach [2].

In this context, a control problem is now formulated as follows. Assume that the *plant*, a dynamical system  $\Sigma_p = (T, W_1 \times C, \mathfrak{B}_p)$  is given. The signal space of the plant is given as a Cartesian product, where the second factor,  $C$ , denotes the space in which  $c$ , the interconnection variable, takes its values. Consider now a family  $\mathfrak{C}$  of dynamical systems, all with common time axis  $T$ . We also assume that the elements of  $\mathfrak{C}$  all have the signal space  $C$  in common. An element  $\Sigma_c = (T, C, \mathfrak{B}_c)$  of  $\mathfrak{C}$  is called an *admissible controller*. The interconnected system  $\Sigma_p \wedge \Sigma_c$  is called the *controlled system*. The control problem for the plant  $\Sigma_p$  is now, to specify the set  $\mathfrak{C}$  of admissible controllers, to describe what desirable properties the controlled system should have, and finally, to find an admissible controller  $\Sigma_c$  such that  $\Sigma_p \wedge \Sigma_c$  has the desired properties.

### 3 Linear time-invariant differential systems

In this paper we restrict ourselves to systems described by linear differential equations with constant coefficients. Let  $\xi$  denote an indeterminate, and let  $\mathbb{R}^{* \times q}[\xi]$  be the set of all real polynomial matrices with  $q$  columns and any (finite) number of rows. An element  $R \in \mathbb{R}^{* \times q}[\xi]$  can be written explicitly as  $R(\xi) = R_0 + R_1\xi + R_2\xi^2 + \dots + R_N\xi^N$ , for given real matrices  $R_0, R_1, \dots, R_N$ . Consider now the system of differential equations

$$R_0 w + R_1 \frac{dw}{dt} + \dots + R_N \frac{d^N w}{dt^N} = 0,$$

or, in compact notation,

$$R\left(\frac{d}{dt}\right)w = 0. \quad (3.1)$$

We will deal with systems  $\Sigma = (\mathbb{R}, \mathbb{R}^q, \mathfrak{B})$  with time axis  $\mathbb{R}$ , signal space  $\mathbb{R}^q$ , and behavior  $\mathfrak{B}$  equal to the solution set of 3.1. Of course, we should make precise what we mean by a function  $w : \mathbb{R} \rightarrow \mathbb{R}^q$  to be a solution of 3.1. We call  $w : \mathbb{R} \rightarrow \mathbb{R}^q$  a  *$C^\infty$ -solution* of 3.1 if  $w$  is infinitely often differentiable and if

$$R_0 w(t) + R_1 \frac{dw}{dt}(t) + \dots + R_N \frac{d^N w}{dt^N}(t) = 0$$

holds for all  $t \in \mathbb{R}$ . For our purposes, the notion of  $C^\infty$ -solution is too restrictive, because, for example, it excludes the possibility for a signal  $w$  to be discontinuous. Therefore we also consider the notion of *weak solution*: a function  $w : \mathbb{R} \rightarrow \mathbb{R}^q$  will be called a weak solution of 3.1 if  $w \in L^{loc}(\mathbb{R}, \mathbb{R}^q)$  and if it satisfies the differential equation 3.1 in distributional sense, i.e.,  $\int \langle R^T(-\frac{d}{dt})f(t), w(t) \rangle dt = 0$  for all  $f \in C^\infty(\mathbb{R}, \mathbb{R}^q)$  of

compact support. Here,  $\langle \cdot, \cdot \rangle$  denotes the standard inner product on  $\mathbb{R}^q$ .

We define the behavior of 3.1 in terms of its weak solutions: 3.1 defines the dynamical system  $\Sigma(R) = (\mathbb{R}, \mathbb{R}^q, \mathfrak{B}(R))$  with  $\mathfrak{B}(R)$  the set of weak solutions of 3.1. It can be shown that  $\mathfrak{B}(R) \cap C^\infty(\mathbb{R}, \mathbb{R}^q)$  is dense (in the topology of  $L^{loc}(\mathbb{R}, \mathbb{R}^q)$ ) in  $\mathfrak{B}(R)$ . Thus, every weak solution of 3.1 can actually be approximated by a  $C^\infty$  one. This means that intuitively we can usually think of  $\mathfrak{B}(R)$  as simply consisting of the set of  $C^\infty$  solutions of 3.1.

The set of all systems obtained in this way is denoted by  $\mathcal{L}^q$ . A representation of the system  $\Sigma \in \mathcal{L}^q$  in terms of a differential equation 3.1 is called a *kernel representation* of  $\Sigma$ .

We will now recall the notion of controllability. The dynamical system  $\Sigma$  is called *controllable* if its behavior  $\mathfrak{B}$  has the property that for any pair of trajectories  $w_1, w_2 \in \mathfrak{B}$ , there exists a trajectory  $w \in \mathfrak{B}$  and  $\Delta \geq 0$  such that  $w(t) = w_1(t)$  for  $t < 0$  and  $w(t + \Delta) = w_2(t)$  for  $t > 0$ . The system with kernel representation 3.1 is controllable if and only if  $\text{rank}(R(\lambda)) = \text{rank}(R)$  for all  $\lambda \in \mathbb{C}$ , i.e., the complex matrix  $R(\lambda)$  has constant rank for all  $\lambda$ .

A system  $\Sigma = (\mathbb{R}, \mathbb{R}^q, \mathfrak{B}) \in \mathcal{L}^q$  is said to be *autonomous* if  $((w_1, w_2 \in \mathfrak{B}) \wedge (w_1(t) = w_2(t) \text{ for } t < 0)) \Rightarrow (w_1 = w_2)$ , in other words, if the past of a trajectory in  $\mathfrak{B}$  uniquely defines its future.

For a given  $\Sigma = (\mathbb{R}, \mathbb{R}^q, \mathfrak{B}) \in \mathcal{L}^q$ ,  $p(\Sigma)$  will denote the number of output components of  $\Sigma$ . This number is equal to  $\text{rank}(R)$ , with  $R$  any polynomial matrix such that  $\mathfrak{B} = \mathfrak{B}(R)$ . It can be shown that  $\Sigma = (\mathbb{R}, \mathbb{R}^q, \mathfrak{B}) \in \mathcal{L}^q$  is autonomous if and only if  $p(\Sigma) = q$ , i.e., there exists  $R \in \mathbb{R}^{q \times q}[\xi]$  with  $\det(R) \neq 0$  such that  $\mathfrak{B}(R) = \mathfrak{B}$ .

Assume now that  $\Sigma \in \mathcal{L}^q$  is autonomous and that  $\mathfrak{B} = \mathfrak{B}(R)$  with  $R \in \mathbb{R}^{q \times q}[\xi]$ . Obviously, for any non-singular diagonal matrix  $\alpha \in \mathbb{R}^{q \times q}$ ,  $\mathfrak{B}(R) = \mathfrak{B}(\alpha R)$ . Therefore we can always choose  $R$  such that  $\det(R)$  is a monic polynomial. We will denote this polynomial by  $\chi_\Sigma$  and call it the *characteristic polynomial* of  $\Sigma$ . It can be seen that  $\chi_\Sigma$  depends only on  $\Sigma \in \mathcal{L}^q$  (and not on the matrix polynomial  $R$  which we have used to define it).

A polynomial  $p \in \mathbb{R}[\xi]$  is called a *Hurwitz polynomial* if  $p \neq 0$  and if it has all its roots in the open left half of the complex plane. Similarly we will call  $R \in \mathbb{R}^{q \times q}[\xi]$  *Hurwitz* if  $\det(R)$  is.

Assume that  $\Sigma = (\mathbb{R}, \mathbb{R}^q, \mathfrak{B}) \in \mathcal{L}^q$  is autonomous. We will call  $\Sigma$  *stable* if  $w \in \mathfrak{B}$  implies  $\lim_{t \rightarrow \infty} w(t) = 0$ . (Often this would be called *asymptotic stability* but, in keeping with usage which has become customary, we will simply refer to this property as *stability*).

We need a couple of minor refinements, related to controllability, before embarking on control questions. Let  $\Sigma \in \mathcal{L}^q$ . Then, as we have just seen,  $\Sigma$  is controllable if and only if  $\text{rank}(R(\lambda))$  is constant for  $\lambda \in \mathbb{C}$ . The set

$$\Lambda(\Sigma) = \{\lambda \in \mathbb{C} \mid \text{rank}(R(\lambda)) < \text{rank}(R)\}$$

is called the set of *uncontrollable exponents* of  $\Sigma$ . They

play the role of the *uncontrollable modes* in state space systems. More generally, assume that  $R$  is minimal, i.e., the number of rows of  $R$  is equal to  $p(\Sigma)$ . Then it can be factored as  $R = FR'$  with  $\mathfrak{B}(R') \in \mathcal{L}^q$  controllable, and  $F \in \mathbb{R}^{p(\Sigma) \times p(\Sigma)}[\xi]$  having  $\det(F) \neq 0$ . Obviously, we can assume that  $\det(F)$  is monic. It can be shown that  $\det(F)$  depends on  $\Sigma$  only. We will call it the *characteristic polynomial of the uncontrollable part of  $\Sigma$*  and denote it as  $X_{\Sigma^{uc}}$ . This nomenclature can be justified as follows. Let  $\Sigma = (\mathbb{R}, \mathbb{R}^q, \mathfrak{B}) \in \mathcal{L}^q$ . Then there exists  $\Sigma_1 = (\mathbb{R}, \mathbb{R}^q, \mathfrak{B}_1) \in \mathcal{L}^q$  and  $\Sigma_2 = (\mathbb{R}, \mathbb{R}^q, \mathfrak{B}_2) \in \mathcal{L}^q$  such that (i)  $\Sigma_1$  is controllable, (ii)  $\Sigma_2$  is autonomous, and (iii)  $\mathfrak{B} = \mathfrak{B}_1 \oplus \mathfrak{B}_2$ . It can be shown that  $\Sigma_1$  (called the *controllable part* of  $\Sigma$ ) is uniquely defined by  $\Sigma$ . However, whereas  $\Sigma_2$ , the uncontrollable part, is not uniquely defined by  $\Sigma$ ,  $X_{\Sigma_2}$  is. In terms of  $\Sigma$ , we have  $X_{\Sigma_2} = X_{\Sigma^{uc}}$ . A refinement of the notion of controllability is that of stabilizability. We will call  $\Sigma = (\mathbb{R}, \mathbb{R}^q, \mathfrak{B}) \in \mathcal{L}^q$  *stabilizable* if for each  $w \in \mathfrak{B}$  there exists  $w' \in \mathfrak{B}$  such that  $w'(t) = w(t)$  for  $t < 0$  and such that  $\lim_{t \rightarrow \infty} w'(t) = 0$ . It is easy to prove that  $\Sigma(R)$  is stabilizable if and only if  $\text{rank}(R(\lambda)) = \text{rank}(R)$  for all  $\lambda \in \mathbb{C}$  such that  $\text{Re}(\lambda) \geq 0$ , in other words, the uncontrollable exponents of  $\Sigma(R)$  must have negative real parts, equivalently,  $X_{\Sigma^{uc}}$  the characteristic polynomial of the uncontrollable part, must be Hurwitz.

## 4 Pole placement and stabilization in a behavioral framework.

In this section we will study our first control problem, with control viewed as interconnection as explained in section 2. The plant is a given dynamical system  $\Sigma \in \mathcal{L}^q$ . We will assume that the controller (and hence the controlled system) is also a linear differential system. Let  $\Sigma_k = (\mathbb{R}, \mathbb{R}^q, \mathfrak{B}_k) \in \mathcal{L}^q$ ,  $k = 1, 2$ . We will call  $\Sigma_2$  a *subsystem* of  $\Sigma_1$  (denoted  $\Sigma_2 < \Sigma_1$ ) if  $\mathfrak{B}_2 \subseteq \mathfrak{B}_1$ . It is proven in [ ] that if  $\Sigma_k = \mathfrak{B}(R_k)$ , then  $\Sigma_2 < \Sigma_1$  if and only if there exists  $F_2 \in \mathbb{R}^{\bullet \times \bullet}[\xi]$  such that  $R_1 = F_2 R_2$ . Obviously, for any  $\Sigma, \Sigma' \in \mathcal{L}^q$ ,  $\Sigma \wedge \Sigma'$  will be a subsystem of  $\Sigma$ . Our first result is the analogue of the classical pole placement result.

**Theorem 4.1 :** *Let  $\Sigma \in \mathcal{L}^q$ , and assume that  $\Sigma$  is not autonomous. Then for any monic  $r \in \mathbb{R}[\xi]$  there exists  $\Sigma' \in \mathcal{L}^q$  such that  $\chi_{\Sigma \wedge \Sigma'} = r$ . If  $\Sigma \in \mathcal{L}^q$  is autonomous then there exists  $\Sigma' \in \mathcal{L}^q$  such that  $\chi_{\Sigma \wedge \Sigma'} = r$  if and only if  $r$  is a factor of  $\chi_\Sigma$ .*

This result guarantees pole placement (and hence stabilizability) for any  $\Sigma \in \mathcal{L}^q$  which is not autonomous, i.e., as long as in the system of differential equation (1) describing  $\Sigma$  there are less equations than variables  $p(\Sigma) < q$ . This means that at least one of the variables  $w_1, w_2, \dots, w_q$  is an input variable. Note that not even controllability or stabilizability of  $\Sigma$  is required! In

particular stabilizability thus holds by simple interconnection, regardless of the location of the uncontrollable exponents of  $\Sigma$ . This result is due to the fact that the class of admissible controllers was chosen to be all of  $\mathcal{L}^q$ . In particular by taking  $\Sigma' = (\mathbb{R}, \mathbb{R}^q, 0)$  stability is trivially obtained. We will return to the question how such a control law could be implemented.

## 5 Regular interconnection.

We will now introduce an important type of interconnection (which, as was shown in [2], corresponds to singular feedback). Let  $\Sigma, \Sigma' \in \mathcal{L}^q$ . We will call  $\Sigma \wedge \Sigma'$  a *regular interconnection* if

$$p(\Sigma \wedge \Sigma') = p(\Sigma) + p(\Sigma')$$

There are a number of alternative equivalent ways of expressing this. First, if  $\Sigma = \ker R(\frac{d}{dt})$  and  $\Sigma' = \Sigma(R')$ , with  $R$  and  $R'$  of full row rank, then  $\Sigma \wedge \Sigma'$  is a regular interconnection if and only if  $\begin{bmatrix} R \\ R' \end{bmatrix}$  is also a full row rank polynomial matrix.

It is trivial to see that any subsystem  $\Sigma''$  of  $\Sigma$  can be realized through interconnection. Indeed if we take  $\Sigma' = \Sigma''$  then obviously  $\Sigma \wedge \Sigma' = \Sigma''$ . However, this interconnection is regular only in the trivial case  $p(\Sigma) = 0$ . The question thus arises when  $\Sigma''$  can be achieved by regular interconnection. Actually, we shall now see that any subsystem of  $\Sigma$  can still be realized through regular interconnection provided that  $\Sigma$  is controllable!

**Theorem 5.1 :** *Assume that  $\Sigma \in \mathcal{L}^q$  is controllable. Let  $\Sigma'' \in \mathcal{L}^q$  be a subsystem of  $\Sigma$ . Then there exists a  $\Sigma' \in \mathcal{L}^q$  such that  $\Sigma \wedge \Sigma' = \Sigma''$  and such that this interconnection is regular.*

The important conclusion which may be drawn from the above theorem will be that singular feedback control problems for controllable systems amounts to looking for a suitable subsystem.

One important variation of the above theorem worth stating is the following.

**Theorem 5.2 :** *Assume that  $\Sigma \in \mathcal{L}^q$  and let  $r \in \mathbb{R}[\xi]$  be monic. Then there exists  $\Sigma' \in \mathcal{L}^q$  such that  $\Sigma \wedge \Sigma'$  is (i) a regular interconnection and (ii)  $\chi_{\Sigma \wedge \Sigma'} = r$  if and only if  $\chi_\Sigma^{uc}$  is a factor of  $r$ .*

In particular, Theorem 5.2 implies that there exists a  $\Sigma'$  such that  $\Sigma \wedge \Sigma'$  is (i) a regular interconnection and (ii) stable if and only if  $\Sigma$  is stabilizable.

## 6 Linear differential systems with disturbances

In problems of pole-placement and stabilization, we will typically look for controllers that make the controlled system autonomous. Thus, a controller  $\Sigma_c$  will be admissible

only if the controlled system  $\Sigma_p \wedge \Sigma_c$  is autonomous, i.e.,  $p(\Sigma_p \wedge \Sigma_c) = q$ .

In many control problems, in the plant  $\Sigma_p$  to be controlled, some components of the manifest variable will play the role of *unknown disturbances* and other components will play the role of *variables to be kept small*.

In such cases, our starting point is that the manifest variable  $w$  of the plant  $\Sigma_p$  consist of three components,  $w = (z, d, c)$ . Here,  $z$  is the signal that we want to keep small and  $d$  is the disturbance. Finally,  $c$  is the interconnection variable as referred to in section 2. Accordingly, the signal space of  $\Sigma_p$  is equal to the Cartesian product  $Z \times D \times C$ , with  $Z$ ,  $D$ , and  $C$  sets in which  $z$ ,  $d$ , and  $c$  take their values, respectively.

The component  $d$  is interpreted as an unknown disturbance. On a set-theoretic level, this can be formalized by assuming that *any* function  $d : T \rightarrow D$  can occur as the second component of the signal vector  $w$  of  $\Sigma_p$ . In order to formalize this, if, in general, we have a dynamical system  $\Sigma = (T, W_1 \times W_2, \mathcal{B})$ , with manifest variable  $(w_1, w_2)$  and if  $\pi : W_1 \times W_2 \rightarrow W_2$  is the projection  $\pi(w_1, w_2) = w_2$ , then the variable  $w_2$  is called *free* if  $\pi(\mathcal{B}) = W_2^T$ . In  $\Sigma_p$ ,  $d$  is thus assumed to be free.

Now, given any controller, in the controlled system,  $d$  is still interpreted as an unknown disturbance. Hence, again, *any*  $d$  should be possible as the second component of the signal vector  $(z, d)$  of the controlled system. If this requirement holds, then we call the controller admissible:  $\Sigma_c$  is admissible if in the controlled system  $\Sigma_p \wedge \Sigma_c$  the variable  $d$  is free. More precisely, consider any dynamical system  $\Sigma_c = (T, C, \mathcal{B}_c)$ , with the same time axis as the plant  $\Sigma_p$ , and whose signal space is equal to the interconnection space  $C$  of  $\Sigma_p$ . According to the above definition, the interconnection is  $\Sigma_p \wedge \Sigma_c = (T, Z \times D, \mathcal{B})$ , with  $\mathcal{B} = \{(z, d) : T \rightarrow Z \times D \mid \text{there exists } c \in \mathcal{B}_c \text{ such that } (z, d, c) \in \mathcal{B}_p\}$ . The controller  $\Sigma_c$  is called admissible if  $\pi(\mathcal{B}) = D^T$ .

Typically, in the controlled system we want the signal  $z$  to be small, regardless of the disturbance  $d$  that occurs. This specification can of course be formalized in many ways. One possibility is to assume that  $T$ , the time axis, is equal to  $\mathbb{R}$  and that the signal spaces  $Z$ ,  $D$  and  $C$  are finite dimensional Euclidean spaces. The size of the signals  $z$  and  $d$  can, for example, be measured by their quadratic integrals  $\|z\|_2^2 = \int \|z(t)\|^2 dt$  and  $\|d\|_2^2 = \int \|d(t)\|^2 dt$ , where the integrals range over  $\mathbb{R}$ . For a given finite dimensional Euclidean space  $X$ , let  $L_2(\mathbb{R}, X)$  be the space of all functions  $f$  from  $\mathbb{R}$  to  $X$  for which  $\int \|f(t)\|^2 dt$  is finite.

The  $H_\infty$  performance of the controlled system  $\Sigma_p \wedge \Sigma_c$  is defined as

$$J(\Sigma_c) := \inf \{ \gamma \geq 0 \mid \|z\|_2 \leq \gamma \|d\|_2 \text{ for all } (z, d) \in \mathcal{B} \cap L_2(\mathbb{R}, Z \times D) \}.$$

The  $H_\infty$  optimal control problem is to minimize  $J(\Sigma_c)$  over all admissible controllers  $\Sigma_c$ . Of course, sometimes it makes more sense to measure the size of the signals

appearing in the system using  $L_1$ - norms. This leads to the  $L_1$ -optimal control problem in a behavioral setting. The issue of stability can be incorporated in this context by defining an admissible controller  $\Sigma_c$  to be a *stabilizing controller* if in the controlled system the signal  $z$  converges to zero whenever  $d = 0$ , i.e., if  $(z, 0) \in \mathcal{B}$  implies that  $\lim_{t \rightarrow \infty} z(t) = 0$ .

In this section we deal with differential systems whose manifest variable  $w$  consists of three components,  $w = \text{col}(z, d, c)$ . Let  $\Sigma_p \in \mathcal{L}^q$  (the plant) be such a system. We assume that  $z$ ,  $d$ , and  $c$  take their values in  $\mathbb{R}^z$ ,  $\mathbb{R}^d$ , and  $\mathbb{R}^c$  respectively, so the signal space of  $\Sigma_p$  equals  $\mathbb{R}^q = \mathbb{R}^z \times \mathbb{R}^d \times \mathbb{R}^c$ . A standing assumption will be that the system  $\Sigma_p$  is controllable. It was shown in [2] that such system admits an image representation

$$w = W\left(\frac{d}{dt}\right)\ell$$

for some real polynomial matrix  $W$ , say with  $m$  columns. Without loss of generality, we assume moreover that this image representation is observable, i.e., that  $W(\lambda)$  has full column rank  $m$  for all  $\lambda \in \mathbb{C}$ . According to the partition of  $w$  into  $\text{col}(z, d, c)$ , we partition

$$W = \begin{pmatrix} Z \\ D \\ C \end{pmatrix}, \quad (6.1)$$

with  $Z$ ,  $D$ , and  $C$  real polynomial matrices of appropriate dimensions. The behavior  $\mathcal{B}_p$  of  $\Sigma_p$  therefore consists of signals  $w = \text{col}(z, d, c) \in L^{loc}(\mathbb{R}, \mathbb{R}^z \times \mathbb{R}^d \times \mathbb{R}^c)$  for which there exists a function  $\ell \in L^{loc}(\mathbb{R}, \mathbb{R}^1)$  such that  $z = Z(\frac{d}{dt})\ell$ ,  $d = D(\frac{d}{dt})\ell$ , and  $c = C(\frac{d}{dt})\ell$ . Recall that the signal  $d$  is interpreted as an unknown disturbance. We have formalized this by assuming that  $d$  is free. In the present context of linear differential systems we formalize this as follows. In general, if we have a dynamical system  $\Sigma \in \mathcal{L}^q$ ,  $\Sigma = (\mathbb{R}, \mathbb{R}^{q_1} \times \mathbb{R}^{q_2}, \mathcal{B})$ , with manifest variable  $(w_1, w_2)$  and if  $\pi : \mathbb{R}^{q_1} \times \mathbb{R}^{q_2} \rightarrow \mathbb{R}^{q_2}$  is the projection  $\pi(w_1, w_2) = w_2$ , then the variable  $w_2$  is called  *$C^\infty$ -free* if  $\pi(\mathcal{B} \cap C^\infty(\mathbb{R}, \mathbb{R}^{q_1} \times \mathbb{R}^{q_2})) = C^\infty(\mathbb{R}, \mathbb{R}^{q_2})$ . Equivalently: if every  $C^\infty$  function can occur as the second component of a  $C^\infty$  trajectory  $(w_1, w_2)$  of  $\Sigma$ .

Let us now consider how this notion translates as a property of an image representation. If  $\Sigma$  is given in image representation,

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} W_1(\frac{d}{dt}) \\ W_2(\frac{d}{dt}) \end{pmatrix} \ell,$$

then  $w_2$  is  $C^\infty$ -free if and only if the differential operator  $W_2(\frac{d}{dt}) : C^\infty(\mathbb{R}, \mathbb{R}^m) \rightarrow C^\infty(\mathbb{R}, \mathbb{R}^{q_2})$  is surjective. This is the case if and only if the polynomial matrix  $W_2$  has full row rank. This equivalence is easily proven, for example, via the Smith form of  $W_2$ .

Thus, we assume that, in our plant  $\Sigma_p$ , the variable  $d$  is  $C^\infty$ -free. Hence, in 6.1 we will assume that the polynomial matrix  $D$  has full row rank  $d$ , equivalently, that the differential operator  $D(\frac{d}{dt})$  is surjective.

We will now specify the set of admissible controllers in the context of linear differential systems. In principle, any linear differential system  $\Sigma_c = (\mathbb{R}, \mathbb{R}^c, \mathfrak{B}_c)$  with manifest variable  $c$  and signal space equal to the interconnection space  $\mathbb{R}^c$  of the plant  $\Sigma_p$  is a candidate admissible controller. However, for obvious reasons, we will require that in the interconnected system  $\Sigma_p \wedge \Sigma_c$ , the variable  $d$  should still be free. In the context of linear differential systems we will interpret this in the sense that  $d$  should remain  $C^\infty$ -free: the linear differential system  $\Sigma_c = (\mathbb{R}, \mathbb{R}^c, \mathfrak{B}_c)$  is called an *admissible controller* for our plant  $\Sigma_p$  if in  $\Sigma_p \wedge \Sigma_c$  the variable  $d$  is  $C^\infty$ -free.

Let us study how this requirement translates into a property of a kernel representation of the controller  $\Sigma_c$ . Suppose that  $K$  is a real polynomial matrix such that  $K(\frac{d}{dt})c = 0$  is a kernel representation of  $\Sigma_c$ .

It is easily seen that the condition that in the interconnected system  $d$  is  $C^\infty$ -free is equivalent to the requirement that for all  $d \in C^\infty(\mathbb{R}, \mathbb{R}^d)$  there exists  $\ell \in C^\infty(\mathbb{R}, \mathbb{R}^1)$  such that  $d = D(\frac{d}{dt})\ell$  and  $K(\frac{d}{dt})C(\frac{d}{dt})\ell = 0$ , or, equivalently,  $D(\frac{d}{dt})\ker K(\frac{d}{dt})C(\frac{d}{dt}) = C^\infty(\mathbb{R}, \mathbb{R}^d)$ . The following lemma shows how to translate this condition in terms of a rank condition on the polynomial matrices defining the system and the controller:

**Proposition 6.1 :** The controller  $\Sigma_c$  with kernel representation  $K(\frac{d}{dt})c = 0$  is admissible if and only if

$$\text{rank} \begin{pmatrix} D \\ KC \end{pmatrix} = d + \text{rank } KC. \quad (6.2)$$

## 7 Full information control problems

In this section we will explain what we mean by a control problem to be a *full information* problem.

In general, if  $\Sigma = (T, W_1 \times W_2, \mathcal{B})$  is a dynamical system with manifest variable  $w = \text{col}(w_1, w_2)$ , then we call  $w_1$  *observable from  $w_2$*  if  $w_1$  is completely determined by  $w_2$ , in the sense that if  $\text{col}(w_1^1, w_2^1)$  and  $\text{col}(w_1^2, w_2^2)$  are in  $\mathcal{B}$  and if  $w_2^1 = w_2^2$ , then  $w_1^1 = w_1^2$ . If  $w_1$  is observable from  $w_2$  then we call  $w_2$  a *full information variable* for  $\Sigma$ : in this case the whole manifest variable  $w$  can be determined from the component  $w_2$  alone.

Consider now, as before, a plant  $\Sigma_p = (T, Z \times D \times C, \mathfrak{B}_p)$  with manifest variable  $w = \text{col}(z, d, c)$ , time axis  $T = \mathbb{R}$  and where  $Z$ ,  $D$ , and  $C$  are Euclidean spaces. If the interconnection variable  $c$  is a full information variable for  $\Sigma_p$ , then we call the corresponding control problem a *full information problem*.

We will now investigate how the property that  $c$  is a full information variable translates to the situation that our plant is a linear differential system in image representation.

Suppose first, in general, that we have a linear differential system  $\Sigma$  given by an observable image representation

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} W_1(\frac{d}{dt}) \\ W_2(\frac{d}{dt}) \end{pmatrix} \ell.$$

We claim that  $w_2$  is a full information variable for  $\Sigma$  if and only if the system  $w_2 = W_2(\frac{d}{dt})\ell$  is observable, equivalently,  $W_2(\lambda)$  has full column rank for all  $\lambda \in \mathbb{C}$ . Indeed, let  $W_2(\frac{d}{dt})\ell_1 = W_2(\frac{d}{dt})\ell_2$ . Then, by observability of  $w_1$  from  $w_2$ , we must have  $W_1(\frac{d}{dt})\ell_1 = W_1(\frac{d}{dt})\ell_2$ . Since the representation itself is observable, we conclude that  $\ell_1 = \ell_2$ . The converse is immediate.

Consider now our plant  $\Sigma_p = (\mathbb{R}, \mathbb{R}^Z \times \mathbb{R}^D \times \mathbb{R}^C, \mathfrak{B}_p)$  with image representation

$$\begin{pmatrix} z \\ d \\ c \end{pmatrix} = \begin{pmatrix} Z(\frac{d}{dt}) \\ D(\frac{d}{dt}) \\ C(\frac{d}{dt}) \end{pmatrix} \ell. \quad (7.1)$$

Then we arrive at the following:

**Proposition 7.1 :** Assume that the representation 7.1 is observable. Then the interconnection variable  $c$  is a full information variable for  $\Sigma_p$  if and only if  $c = C(\frac{d}{dt})\ell$  is an observable image representation, equivalently,  $\text{rank } C(\lambda) = m$  for all  $\lambda \in \mathbb{C}$ .

Let  $\Sigma_c = (\mathbb{R}, \mathbb{R}^c, \mathfrak{B}_c)$  be a controller with kernel representation  $K(\frac{d}{dt})c = 0$ . Let  $\bar{c}$  and  $\bar{\ell}$  be related by  $\bar{c} = C(\frac{d}{dt})\bar{\ell}$ . Clearly, if we define  $K' := KC$  then  $K(\frac{d}{dt})\bar{c} = 0$  iff  $K'(\frac{d}{dt})\bar{\ell} = 0$ . Thus, if we define a new controller by

$$\mathfrak{B}'_c := \{c \in \mathcal{L}^{loc}(\mathbb{R}, \mathbb{R}^c) \mid \text{there exists } \ell \in \mathcal{L}^{loc}(\mathbb{R}, \mathbb{R}^1) \text{ such that } K'(\frac{d}{dt})\ell = 0 \text{ and } c = C(\frac{d}{dt})\ell\} \quad (7.2)$$

then clearly the controlled systems  $\Sigma_p \wedge \Sigma_c$  and  $\Sigma_p \wedge \Sigma'_c$  are equal. In other words, given a controller with representation  $K(\frac{d}{dt})c = 0$ , there exists a controller with representation  $c = C(\frac{d}{dt})\ell$ ,  $K'(\frac{d}{dt})\ell = 0$  that yields the same controlled system.

If  $c = C(\frac{d}{dt})\ell$  is observable, then also the converse holds, i.e., for every controller of the form  $c = C(\frac{d}{dt})\ell$ ,  $K'(\frac{d}{dt})\ell = 0$  there exists a controller of the form  $K(\frac{d}{dt})c = 0$  yielding the same controlled system. Indeed, in that case there exists a polynomial left-inverse  $L$  of  $C$ , so if we have a controller given by 7.2, and if we define  $K := K'L$ , then for all  $\bar{c}$  and  $\bar{\ell}$  related by  $\bar{c} = C(\frac{d}{dt})\bar{\ell}$ , we have  $K(\frac{d}{dt})\bar{c} = 0$  iff  $K'(\frac{d}{dt})\bar{\ell} = 0$ . This means that the controller  $\Sigma'_c = (\mathbb{R}, \mathbb{R}^c, \mathfrak{B}'_c)$  and the controller  $\Sigma_c$  with kernel representation  $K(\frac{d}{dt})c = 0$  yield the same controlled system.

Summarizing, if in the plant  $\Sigma_p$  the representation  $c = C(\frac{d}{dt})\ell$  is observable, i.e., in the full information case, the set of controllers of the form

$$c = C(\frac{d}{dt})\ell, \quad K'(\frac{d}{dt})\ell = 0 \quad (7.3)$$

and the set of controllers of the form

$$K(\frac{d}{dt})c = 0 \quad (7.4)$$

yields one and the same set of controlled systems. Therefore, we may without loss of generality restrict ourselves to the set of all controllers given by 7.3 (with behavior given by (7.2)), where  $K'$  ranges over the set of all polynomial matrices with 1 columns.

Without loss of generality we can also restrict ourselves to polynomial matrices  $K'$  with full row rank. In the following lemma we deal with the question under what conditions a controller (7.2) is admissible:

**Lemma 7.2 :** *Consider the plant  $\Sigma_p$  with observable image representation (7.1). Assume that  $c$  is a full information variable. Then the controller (7.2) with  $K'$  of full row rank is admissible if and only if  $\begin{pmatrix} D \\ K' \end{pmatrix}$  has full row rank.*

In the sequel we will simply write  $K$  instead of  $K'$  and  $\mathfrak{B}_c$  instead of  $\mathfrak{B}'_c$ .

To summarize, for the plant  $\Sigma_p$  given by the observable image representation (7.1), and with  $c$  a full information variable, we can consider controllers  $\Sigma_c$  given by

$$c = C\left(\frac{d}{dt}\right)\ell, \quad K\left(\frac{d}{dt}\right)\ell = 0 \quad (7.5)$$

If  $K$  has full row rank then such a controller is admissible iff

$$\begin{pmatrix} D \\ K \end{pmatrix} \text{ has full row rank} \quad (7.6)$$

Note that if  $\Sigma_c$  is admissible and  $K$  has full row rank, then  $K$  has at most  $1 - d$  rows. These observations are pursued further in [7], where a detailed study is made of the full information  $H_\infty$ -problem.

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